DERIVATION OF ENGINEERING-RELEVANT DEFORMATION PARAMETERS FROM REPEATED SURVEYS OF SURFACE-LIKE CONSTRUCTIONS

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Abstract: The deformation of a surface-like construction is a complicated problem in the deformation analysis of Riemannian manifolds. The relevant mathematical theory and a relatively simple solution algorithm are developed, which allow the computation of invariant deformation parameters from coordinate displacements of isolated control points. The computed parameters are the dilatation and maximum shear strain at each surface point as well as variations in curvatures related to surface bending. The required interpolation of coordinate displacements over the whole surface is realized by stochastic interpolation, i.e. prediction where displacements are modeled as spatially correlated random variables.

1. Introduction

The study of the deformation of constructions by repeated surveys usually results in point displacements between two campaigns, which however provide little direct information to the construction engineer; in order to access the safety of the construction he is rather concerned with stresses related to the strength of the material. Thus instead of point displacements we need rather strain parameters which can be related to stresses through the constitutive equations of the specific material as provided by the theory of elasticity.

Modern architecture resorts many times to curved elements in order to achieve its aesthetic goals. We will be concerned here with curved construction elements with one of their dimensions quite small with the respect to the others, so that they can be virtually studied as curved surface elements. Repeated surveys, e.g. at two epochs t and t', provide corresponding 3-dimensional coordinates $\mathbf{x}_i = \mathbf{x}_i(t) = [x_i \; y_i \; z_i]^T$ and $\mathbf{x}'_i = \mathbf{x}_i(t') = [x'_i \; y'_i \; z'_i]^T$ for a specific set of characteristic points $i = 1, 2, ..., n$. On the other hand any study of deformation calls for continuous knowledge of the deformation mapping $\mathbf{x} \rightarrow \mathbf{x}' = \mathbf{f}(\mathbf{x})$ over the whole surface and an interpolation must take place in order to pass from the discrete information at hand $\mathbf{x}_i \rightarrow \mathbf{x}'_i = \mathbf{f}(\mathbf{x}_i)$ to the continuous one $\mathbf{x}_P \rightarrow \mathbf{x}'_P = \mathbf{f}(\mathbf{x}_P)$ for any other surface point P.

From the point of view the "strength of materials" there are three types of shape alterations in the neighbourhood of any surface point P, with respect to which the material may react in different ways. The first one is dilatation Δ, i.e. local alteration of the surface area (extension for $\Delta > 0$, contraction for $\Delta < 0$ (fig. 1a). The second is shear along any local section of the surface which relates to forces tending to "tear" the surface apart along that direction (fig. 1b).

Of particular importance is to find the direction with the maximum shear strain γ. Finally the third type of deformation relates to the "bending" of the surface, which can be geometrically expresses by the change of curvature. Any plane passing through the surface normal cuts the surface into a curve having curvature the inverse of the radius of the locally best fitting circle.

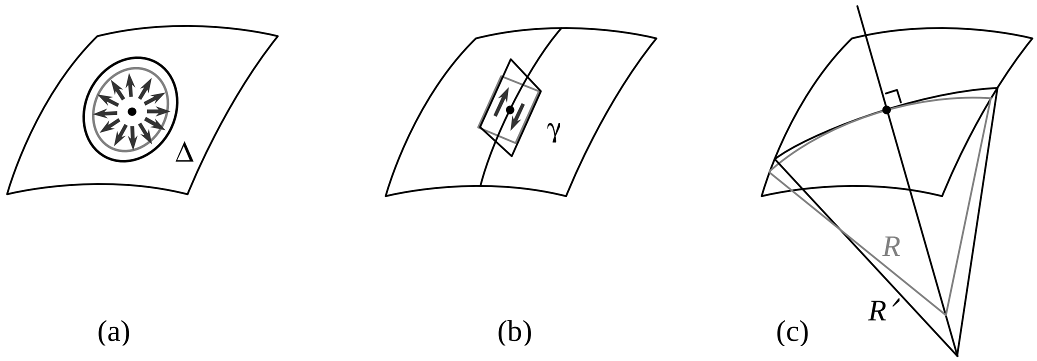

Figure 1: Three modes of surface deformation: (a) dilatation (area variation), (b) maximum shear and (c) bending expressed by the change of the radii of curvature of normal sections

Again we have to determine the direction on the surface where the largest change of curvature and thus bending occurs.
For the first two parameters Δ and γ it is sufficient to study the surface in linear approximation, i.e. to study an infinitesimal part around the point of interest by approximating the surface with a plane. If the surface is described intrinsically by two coordinates $\mathbf{u}=[u_1\ u_2]^T$ it can be represented by an equation of the form $\mathbf{x}=\mathbf{x}(u_1,u_2)=\mathbf{x}(\mathbf{u})$. For the study of surface bending the surface is approximated by a best fitting ellipsoid.
We shall therefore study the deformation of a plane in section 2 and the surface curvatures in various directions in section 3. From the mathematical point of view the present work falls within the deformation analysis of Riemannian manifolds, studied e.g. in relativistic mathematical elasticity (Marsden & Hughes, 1983). For a geodetic application of this topic see Voosoghi (2000).

2. Planar deformation and its application to surface dilatation and shear

We can study the deformation of a plane at a particular point P with the help of the variation of the Cartesian coordinates $\mathbf{x}=\mathbf{x}(t)=[x_1\ x_2]$ and $\mathbf{x}'=\mathbf{x}(t')=[x_1'\ x_2']$ of every surrounding point. This establishes the plane deformation mapping $\mathbf{x}'=\mathbf{f}(\mathbf{x})$ and all local deformation parameters can be derived from the local value of the deformation gradient $\mathbf{F}=\dfrac{\partial \mathbf{x}'}{\partial \mathbf{x}}$. As shown in Dermanis and Kotsakis (2006), and Biagi and Dermanis (2006) we may utilize the eigenvectors and the common eigenvalues of the matrices $\mathbf{F}^T\mathbf{F}$ and $\mathbf{F}\mathbf{F}^T$ in order to express $\mathbf{F}$ through its singular value decomposition (SVD)

$$\mathbf{F}=\mathbf{Q}^T\mathbf{L}\mathbf{P}=\mathbf{R}(-\theta_Q)\mathbf{L}\mathbf{R}(\theta_P)=\begin{bmatrix}\cos\theta_Q & -\sin\theta_Q\\ \sin\theta_Q & \cos\theta_Q\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\cos\theta_P & \sin\theta_P\\ -\sin\theta_P & \cos\theta_P\end{bmatrix}. \qquad (1)$$

Noting that $\mathbf{F}^T\mathbf{F} = \mathbf{P}^T\mathbf{L}^2\mathbf{P}$ and $\mathbf{F}\mathbf{F}^T = \mathbf{Q}^T\mathbf{L}^2\mathbf{Q}$, λ_1^2 and λ_2^2 are the common eigenvalues of the matrices $\mathbf{F}^T\mathbf{F}$ and $\mathbf{F}\mathbf{F}^T$. The corresponding eigenvectors $\mathbf{p}_1$, $\mathbf{p}_2$ of $\mathbf{F}^T\mathbf{F}$ are the columns of the matrix $\mathbf{P}^T = [\mathbf{p}_1\,\mathbf{p}_2]$, while the corresponding eigenvectors $\mathbf{q}_1$, $\mathbf{q}_2$ of $\mathbf{F}\mathbf{F}^T$ are the columns of the matrix $\mathbf{Q}^T = [\mathbf{q}_1\,\mathbf{q}_2]$. Ordering so that $\lambda_1 > \lambda_2$, $\mathbf{p}_1 = [\cos\theta_P \ \sin\theta_P]^T$ is the unit vector in the direction of maximum elongation λ_1 in the epoch t coordinate system, while λ_2 is the minimal elongation in the perpendicular direction. Thus θ_P is the (measured counter-clock-wise) angle from the x_1 axis to the direction of maximum elongation. The same direction at the coordinate system of epoch t' is expressed by the vector $\mathbf{q}_1 = [\cos\theta_Q \ \sin\theta_Q]^T$ and thus θ_Q is the (counter-clock-wise) angle from the x_1' axis to the direction of maximum elongation.

An infinitesimal circle with radius Δr, with area $E = \pi\Delta r^2$ is deformed into an ellipse with axes $\lambda_1\Delta r$, $\lambda_2\Delta r$ having area $E' = \pi\lambda_1\lambda_2\Delta r^2$ and thus $\lambda_1\lambda_2 = E'/E$ is the factor of area increase (or decrease if less than 1). The dilatation

$$\Delta = \lambda_1\lambda_2 - 1 = (E' - E)/E \tag{2}$$

expresses the ratio of area variation $E' - E$ with respect to the original area E.

For the maximum shear strain γ we use (Biagi & Dermanis, 2006) a decomposition $\mathbf{F} = s\mathbf{R}(\psi)\mathbf{\Gamma}_\phi$, where $\mathbf{\Gamma} = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$ is a matrix of shear along the first axis, $\mathbf{\Gamma}_\phi = \mathbf{R}(-\phi)\mathbf{\Gamma}\mathbf{R}(\phi)$ represents shear in the direction angle ϕ , s is a scale parameter and $\mathbf{R}(\psi)$ an additional rotation such that the 4 parameters γ, ϕ , s, ψ are as many as the elements of $\mathbf{F}$. Comparison with the SVD representation leads to the computation of maximum shear and its direction angle ϕ by means of

$$\gamma = \frac{\lambda_1 - \lambda_2}{\sqrt{\lambda_1\lambda_2}}, \qquad \phi = \theta_P - \frac{1}{2}\arctan\frac{2}{-\gamma}. \tag{3}$$

In order to apply the above results for a plane to a curved surface, we need to choose intrinsic coordinates on the surface $\mathbf{u} = [u_1\ u_2]^T$ and $\mathbf{u}' = [u_1'\ u_2']^T$ for both epochs t and t', respectively. For the first epoch t we choose the 3-dimensional horizontal coordinates $u_1 = x$, $u_2 = y$, in which case the surface equation has the form $\mathbf{x} = \mathbf{x}(\mathbf{u}) = [x\ y\ z(x,y)]^T$. For the second epoch t' we choose to identify points on the surface again with their horizontal coordinates they had at time t, i.e. $u_1' = x$, $u_2' = y$. Such coordinates for the deformed shape borrowed from those of the original un-deformed shape are called convected coordinates (see e.g. Marsden & Hughes, 1983, p. 41). The “deformed” surface equation at epoch t' has the form $\mathbf{x}' = \mathbf{x}'(\mathbf{u}') = [x'(x,y)\ y'(x,y)\ z'(x,y)]^T$. The deformation $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is described in this case by the coordinate gradient $\mathbf{F}_\mathbf{u} = \dfrac{\partial\mathbf{u}'}{\partial\mathbf{u}}$, which in the particular case of convected coordina-tes $\mathbf{u}' = \mathbf{u}$, becomes the unit matrix $\mathbf{F}_\mathbf{u} = \mathbf{I}$. It cannot be used for proceeding as in the planar case, because it does not correspond any more to an orthogonal reference system with base vectors of unit length as in the case of planar Cartesian coordinates. Instead it refers to the local reference systems with bases $\mathbf{e}_i = \partial_{u_i}\mathbf{x} = \dfrac{\partial \mathbf{x}}{\partial u_i}$ at epoch t and $\mathbf{e}'_i = \partial_{u'_i}\mathbf{x}' = \dfrac{\partial \mathbf{x}'}{\partial u'_i}$ at epoch t' ($i = 1,2$) formed by the tangent vectors to the surface coordinate lines. Note that plane ($\mathbf{e}_1, \mathbf{e}_2$) represents the planar approximation to the surface at epoch t and the plane ($\mathbf{e}'_1, \mathbf{e}'_2$) does the same for the surface at epoch t'. Specifically

$$\mathbf{e}_1 = \frac{\partial \mathbf{x}}{\partial x} = \begin{bmatrix} 1 & 0 & \dfrac{\partial z}{\partial x} \end{bmatrix}^T, \qquad \mathbf{e}_2 = \frac{\partial \mathbf{x}}{\partial y} = \begin{bmatrix} 0 & 1 & \dfrac{\partial z}{\partial y} \end{bmatrix}^T, \tag{4}$$

$$\mathbf{e}'_1 = \frac{\partial \mathbf{x}'}{\partial x} = \begin{bmatrix} \dfrac{\partial x'}{\partial x} & \dfrac{\partial y'}{\partial x} & \dfrac{\partial z'}{\partial x} \end{bmatrix}^T, \qquad \mathbf{e}'_2 = \frac{\partial \mathbf{x}'}{\partial y} = \begin{bmatrix} \dfrac{\partial x'}{\partial y} & \dfrac{\partial y'}{\partial y} & \dfrac{\partial z'}{\partial y} \end{bmatrix}^T. \tag{5}$$

All we need now is to change the two bases to orthonormal ones $[\tilde{\mathbf{e}}_1\tilde{\mathbf{e}}_2] = [\mathbf{e}_1\mathbf{e}_2]\mathbf{S}$ and $[\tilde{\mathbf{e}}'_1\tilde{\mathbf{e}}'_2] = [\mathbf{e}'_1\mathbf{e}'_2]\mathbf{S}'$, such that $\tilde{\mathbf{e}}_i^T\tilde{\mathbf{e}}_k = \delta_{ik}$ and $\tilde{\mathbf{e}}_i'^T\tilde{\mathbf{e}}'_k = \delta_{ik}$. Thus a local vector $\boldsymbol{\chi}$ at some particular point P, lying within the local tangent plane has the form $\boldsymbol{\chi} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 = [\mathbf{e}_1\,\mathbf{e}_2]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} =$ $= [\mathbf{e}_1\,\mathbf{e}_2]\mathbf{u}$ and will take the new representation $\boldsymbol{\chi} = [\mathbf{e}_1\,\mathbf{e}_2]\mathbf{u} = [\mathbf{e}_1\,\mathbf{e}_2]\mathbf{S}\mathbf{S}^{-1}\mathbf{u} = [\tilde{\mathbf{e}}_1\tilde{\mathbf{e}}_2]\mathbf{S}^{-1}\mathbf{u} =$ $= [\tilde{\mathbf{e}}_1\tilde{\mathbf{e}}_2]\tilde{\mathbf{u}}$ with $\tilde{\mathbf{u}} = \mathbf{S}^{-1}\mathbf{u}$. In a similar way $\boldsymbol{\chi}' = [\mathbf{e}'_1\,\mathbf{e}'_2]\mathbf{u}' = [\tilde{\mathbf{e}}'_1\tilde{\mathbf{e}}'_2]\tilde{\mathbf{u}}'$ with $\tilde{\mathbf{u}}' = \mathbf{S}'^{-1}\mathbf{u}'$ for the epoch t'. Now the planar coordinates $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{u}}'$ refer to orthonormal bases and the gradient we need is $\mathbf{F} = \dfrac{\partial \tilde{\mathbf{u}}'}{\partial \tilde{\mathbf{u}}}$, or in view of the chain rule of differentiation

$$\mathbf{F} = \frac{\partial \tilde{\mathbf{u}}'}{\partial \tilde{\mathbf{u}}} = \frac{\partial \tilde{\mathbf{u}}'}{\partial \mathbf{u}'}\frac{\partial \mathbf{u}'}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \tilde{\mathbf{u}}} = \frac{\partial \tilde{\mathbf{u}}'}{\partial \mathbf{u}'}\mathbf{F}_u\left(\frac{\partial \tilde{\mathbf{u}}}{\partial \mathbf{u}}\right)^{-1} = \mathbf{S}'^{-1}\mathbf{I}(\mathbf{S}^{-1})^{-1} = \mathbf{S}'^{-1}\mathbf{S}. \tag{6}$$

Noting that $[\mathbf{e}_1\mathbf{e}_2] = [\tilde{\mathbf{e}}_1\tilde{\mathbf{e}}_2]\mathbf{S}^{-1}$ and $[\mathbf{e}'_1\mathbf{e}'_2] = [\tilde{\mathbf{e}}'_1\tilde{\mathbf{e}}'_2]\mathbf{S}'^{-1}$, the required transformation matrices $\mathbf{S}$ and $\mathbf{S}'$ can be derived from the metric matrices

$$\tilde{\mathbf{G}} = [\tilde{\mathbf{e}}_1\tilde{\mathbf{e}}_2]^T[\tilde{\mathbf{e}}_1\tilde{\mathbf{e}}_2] = \begin{bmatrix} \tilde{\mathbf{e}}_1^T\tilde{\mathbf{e}}_1 & \tilde{\mathbf{e}}_1^T\tilde{\mathbf{e}}_2 \\ \tilde{\mathbf{e}}_2^T\tilde{\mathbf{e}}_1 & \tilde{\mathbf{e}}_2^T\tilde{\mathbf{e}}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}, \tag{7}$$

$$\mathbf{G} = [\mathbf{e}_1\mathbf{e}_2]^T[\mathbf{e}_1\mathbf{e}_2] = \begin{bmatrix} \mathbf{e}_1^T\mathbf{e}_1 & \mathbf{e}_1^T\mathbf{e}_2 \\ \mathbf{e}_2^T\mathbf{e}_1 & \mathbf{e}_2^T\mathbf{e}_2 \end{bmatrix} = \mathbf{S}^{-T}[\tilde{\mathbf{e}}_1\tilde{\mathbf{e}}_2]^T[\tilde{\mathbf{e}}_1\tilde{\mathbf{e}}_2]\mathbf{S}^{-1} = \mathbf{S}^{-T}\tilde{\mathbf{G}}\mathbf{S}^{-1} = \mathbf{S}^{-T}\mathbf{S}^{-1} \tag{8}$$

and similarly

$$\mathbf{G}' = \begin{bmatrix} \mathbf{e}_1'^T\mathbf{e}'_1 & \mathbf{e}_1'^T\mathbf{e}'_2 \\ \mathbf{e}_2'^T\mathbf{e}'_1 & \mathbf{e}_2'^T\mathbf{e}'_2 \end{bmatrix} = \mathbf{S}'^{-T}\mathbf{S}'^{-1}. \tag{9}$$

If we therefore compute the metric matrices $\mathbf{G}$ and $\mathbf{G}'$ and proceed to their corresponding diagonalizations $\mathbf{G}=\mathbf{R}(-\Theta)\mathbf{M}\mathbf{R}(\Theta)$, $\mathbf{G}'=\mathbf{R}(-\Theta')\mathbf{M}'\mathbf{R}(\Theta')$ we need only rewrite them in the form $\mathbf{G}=\mathbf{R}(\Theta)^T\mathbf{M}^{1/2}\mathbf{M}^{1/2}\mathbf{R}(\Theta)=[\mathbf{M}^{1/2}\mathbf{R}(\Theta)]^T[\mathbf{M}^{1/2}\mathbf{R}(\Theta)]=\mathbf{S}^{-T}\mathbf{S}^{-1}$ to realize that $\mathbf{S}^{-1}=\mathbf{M}^{1/2}\mathbf{R}(\Theta)$, and similarly $\mathbf{S}'^{-1}=\mathbf{M}'^{1/2}\mathbf{R}(\Theta')$ so that

$$\mathbf{S}=\mathbf{R}(-\Theta)\mathbf{M}^{-1/2}, \qquad \mathbf{S}'=\mathbf{R}(-\Theta')\mathbf{M}'^{-1/2} \tag{10}$$

and finally the gradient $\mathbf{F}=\mathbf{S}'^{-1}\mathbf{S}$ for the computation of Δ, γ and ϕ is given by

$$\mathbf{F}=\mathbf{M}'^{1/2}\mathbf{R}(\Theta')\mathbf{R}(-\Theta)\mathbf{M}^{-1/2}=\begin{bmatrix}\sqrt{m_1'/m_1}\cos(\Theta'-\Theta) & \sqrt{m_1'/m_2}\sin(\Theta'-\Theta)\\ -\sqrt{m_2'/m_1}\sin(\Theta'-\Theta) & \sqrt{m_2'/m_2}\cos(\Theta'-\Theta)\end{bmatrix}, \tag{11}$$

where $m_1>m_2$ are the eigenvalues of $\mathbf{G}$ with corresponding eigenvectors $[\cos\Theta\ \ -\sin\Theta]^T$, $[\sin\Theta\ \ \cos\Theta]^T$ and $m_1'>m_2'$ are the eigenvalues of $\mathbf{G}'$ with corresponding eigenvectors $[\cos\Theta'\ \ -\sin\Theta']^T$, $[\sin\Theta'\ \ \cos\Theta']^T$.
Let us recall that for the computation of the metric matrices we need to know the bases $\mathbf{e}_1=[1\ \ 0\ \ \partial_x z]^T$, $\mathbf{e}_2=[0\ \ 1\ \ \partial_y z]^T$, $\mathbf{e}_1'=[\partial_x x'\ \partial_x y'\ \partial_x z']^T$, $\mathbf{e}_2'=[\partial_y x'\ \partial_y y'\ \partial_y z']^T$, and thus the functions $z(x,y)$, $x'(x,y)$, $y'(x,y)$, $z'(x,y)$ which must be derived from the interpolation of the discrete data $[x_i\ \ y_i\ \ z_i]$ and $[x_i'\ \ y_i'\ \ z_i']$.

3. The best fitting ellipsoid to a surface and the corresponding curvatures.

Any surface is an ellipsoid in second order approximation with one axis in the direction of the surface normal defined by the unit vector $\mathbf{n}=|\mathbf{p}|^{-1}\mathbf{p}$ where $\mathbf{p}=\partial_{u_1}\mathbf{x}\times\partial_{u_2}\mathbf{x}=\mathbf{e}_1\times\mathbf{e}_2$. Any plane passing through $\mathbf{n}$ intersects the surface into a "normal section" curve $\mathbf{x}(s)$ having a tangent vector $\mathbf{t}=\partial_s\mathbf{x}=\partial_{\mathbf{u}}\mathbf{x}(d\mathbf{u}/ds)=\partial_{u_1}\mathbf{x}(du_1/ds)+\partial_{u_2}\mathbf{x}(du_2/ds)=[\mathbf{e}_1\mathbf{e}_2]\dot{\mathbf{u}}$, which has components $\dot{\mathbf{u}}\equiv d\mathbf{u}/ds$ in the local basis $[\mathbf{e}_1\mathbf{e}_2]$. The curvature k of such a curve is called a normal curvature and can be expressed by means of the first and second fundamental forms of the surface as (Stoker, 1969)

$$k(\dot{\mathbf{u}})=\frac{II}{I}=\frac{d\mathbf{u}^T\mathbf{L}d\mathbf{u}}{d\mathbf{u}^T\mathbf{G}d\mathbf{u}}=\frac{\dot{\mathbf{u}}^T\mathbf{L}\dot{\mathbf{u}}}{\dot{\mathbf{u}}^T\mathbf{G}\dot{\mathbf{u}}}. \tag{12}$$

The first fundamental form is related to the distance element ds on the surface by $I=ds^2=$ $=d\mathbf{x}^Td\mathbf{x}=(\partial_{\mathbf{u}}\mathbf{x}d\mathbf{u})^T\partial_{\mathbf{u}}\mathbf{x}d\mathbf{u}=d\mathbf{u}^T(\partial_{\mathbf{u}}\mathbf{x})^T\partial_{\mathbf{u}}\mathbf{x}d\mathbf{u}=d\mathbf{u}^T\mathbf{G}d\mathbf{u}$, where $\mathbf{G}=(\partial_{\mathbf{u}}\mathbf{x})^T\partial_{\mathbf{u}}\mathbf{x}=[\mathbf{e}_1\mathbf{e}_2]^T[\mathbf{e}_1\mathbf{e}_2]$ is the already introduced metric matrix $\mathbf{G}$ with elements $g_{ik}=\mathbf{e}_i^T\mathbf{e}_k$. The second fundamental form is $II=-d\mathbf{x}^Td\mathbf{n}=-(\partial_{\mathbf{u}}\mathbf{x}d\mathbf{u})^T\partial_{\mathbf{u}}\mathbf{n}d\mathbf{u}=d\mathbf{u}^T(-\partial_{\mathbf{u}}\mathbf{x})^T\partial_{\mathbf{u}}\mathbf{n}d\mathbf{u}=d\mathbf{u}^T\mathbf{L}d\mathbf{u}$, where $\mathbf{L}=-(\partial_{\mathbf{u}}\mathbf{x})^T\partial_{\mathbf{u}}\mathbf{n}=(\partial^2_{\mathbf{u}}\mathbf{x})^T\mathbf{n}$ is the matrix with elements $L_{ik}=-(\partial_{u_i}\mathbf{x})^T\partial_{u_k}\mathbf{n}=(\partial^2_{u_iu_k}\mathbf{x})^T\mathbf{n}=$ $=(\partial_{u_i}\mathbf{e}_k)^T\mathbf{n}$. The curvature of a normal section undertakes its maximum k_1 and minimum value k_2 (the so called principal curvatures) at two perpendicular to each other "principal" directions $\mathbf{t}_1$ and $\mathbf{t}_2$ (unit tangent vectors to the corresponding normal sections). The principal curvatures are given by

$$k_1 = H + \sqrt{H^2 - K}\,, \qquad k_2 = H - \sqrt{H^2 - K}\,. \tag{13}$$

where

$$H = \frac{1}{2}(k_1 + k_2) = \frac{g_{22}L_{11} - 2g_{12}L_{12} + g_{11}L_{22}}{2\det\mathbf{G}}\,, \qquad K = k_1 k_2 = \frac{\det\mathbf{L}}{\det\mathbf{G}}\,, \tag{14}$$

are the surface mean and Gaussian curvatures, respectively. The principal directions $\mathbf{t}_1$ and $\mathbf{t}_2$ are determined through the corresponding ratios $\rho_i = (\dot{u}_1/\dot{u}_2)_i$ by means of

$$\rho_i = -\frac{L_{12} - k_i g_{12}}{L_{11} - k_i g_{11}} = -\frac{L_{22} - k_i g_{22}}{L_{12} - k_i g_{12}}\,, \tag{15}$$

$$\mathbf{t}_i = \mu_i(\rho_i \mathbf{e}_1 + \mathbf{e}_2) = [\mathbf{e}_1 \ \mathbf{e}_2]\begin{bmatrix} \mu_i \rho_i \\ \mu_i \end{bmatrix}, \qquad i = 1, 2\,. \tag{16}$$

where

$$\mu_i = \frac{1}{\sqrt{g_{11}\rho_i^2 + 2g_{12}\rho_i + g_{22}}}\,. \tag{17}$$

The normal curvature $k(\dot{\mathbf{u}})$ at any direction with unit tangent vector $\mathbf{t}(\dot{\mathbf{u}}) = [\mathbf{e}_1 \mathbf{e}_2]\dot{\mathbf{u}}$ is much easier to compute by means of the angle θ between $\mathbf{t}_1$ and $\mathbf{t}$ ($\mathbf{t}^T\mathbf{t}_1 = \cos\theta$) using "Euler's theorem"

$$k(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta\,. \tag{18}$$

To apply the above results to our problem we need to compute also the normal unit vectors $\mathbf{n} = \dfrac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|}$, $\mathbf{n}' = \dfrac{\mathbf{e}'_1 \times \mathbf{e}'_2}{|\mathbf{e}'_1 \times \mathbf{e}'_2|}$ and the second fundamental form matrices $L_{ik} = (\partial_{u_i}\mathbf{e}_k)^T\mathbf{n}$ and $L'_{ik} = (\partial_{u_i}\mathbf{e}'_k)^T\mathbf{n}'$ at the two epochs t and t', respectively, using the known vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \partial_x z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ z_x \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \partial_y z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ z_y \end{bmatrix}, \quad \mathbf{e}'_1 = \begin{bmatrix} \partial_x x' \\ \partial_x y' \\ \partial_x z' \end{bmatrix} = \begin{bmatrix} x'_x \\ y'_x \\ z'_x \end{bmatrix}, \quad \mathbf{e}'_2 = \begin{bmatrix} \partial_y x' \\ \partial_y y' \\ \partial_y z' \end{bmatrix} = \begin{bmatrix} x'_y \\ y'_y \\ z'_y \end{bmatrix} \tag{19}$$

and their derivatives

$$\partial_{u_1}\mathbf{e}_1 = \partial_x \mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ z_{xx} \end{bmatrix}, \ \partial_{u_2}\mathbf{e}_1 = \partial_y \mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ z_{xy} \end{bmatrix}, \quad \partial_{u_1}\mathbf{e}_2 = \partial_x \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ z_{xy} \end{bmatrix}, \quad \partial_{u_2}\mathbf{e}_2 = \partial_y \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ z_{yy} \end{bmatrix} \tag{20}$$

$$\partial_{u'_1}\mathbf{e}'_1 = \partial_x \mathbf{e}'_1 = \begin{bmatrix} x'_{xx} \\ y'_{xx} \\ z'_{xx} \end{bmatrix}, \quad \partial_{u'_2}\mathbf{e}'_1 = \partial_y \mathbf{e}'_1 = \begin{bmatrix} x'_{xy} \\ y'_{xy} \\ z'_{xy} \end{bmatrix}, \quad \partial_{u'_1}\mathbf{e}'_2 = \partial_x \mathbf{e}'_2 = \begin{bmatrix} x'_{xy} \\ y'_{xy} \\ z'_{xy} \end{bmatrix}, \quad \partial_{u'_2}\mathbf{e}'_2 = \partial_y \mathbf{e}'_2 = \begin{bmatrix} x'_{yy} \\ y'_{yy} \\ z'_{yy} \end{bmatrix}. \quad (21)$$

The unit normal vectors become

$$\mathbf{n} = \frac{1}{\sqrt{{z_x}^2 + {z_y}^2}} \begin{bmatrix} -z_x \\ -z_y \\ 1 \end{bmatrix}, \qquad \mathbf{n}' = \begin{bmatrix} n'_1 \\ n'_2 \\ n'_3 \end{bmatrix} = \frac{1}{D} \begin{bmatrix} y'_x z'_y - z'_x y'_y \\ z'_x x'_y - x'_x z'_y \\ x'_x y'_y - y'_x x'_y \end{bmatrix} \quad (22)$$

where

$$D = |\mathbf{e}'_1 \times \mathbf{e}'_2| = \sqrt{(y'_x z'_y - z'_x y'_y)^2 + (z'_x x'_y - x'_x z'_y)^2 + (x'_x y'_y - y'_x x'_y)^2}\,. \quad (23)$$

The second fundamental form elements are

$$\mathbf{L} = (\partial^2_{\mathbf{u}}\mathbf{x})^T \mathbf{n} = \begin{bmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{bmatrix} = \frac{1}{\sqrt{z_x^2 + z_y^2}} \begin{bmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{bmatrix} \quad (24)$$

$$L'_{11} = (\partial_x \mathbf{e}'_1)^T \mathbf{n}' = x'_{xx} n'_1 + y'_{xx} n'_2 + z'_{xx} n'_3, \qquad L'_{12} = (\partial_x \mathbf{e}'_2)^T \mathbf{n}' = x'_{xy} n'_1 + y'_{xy} n'_2 + z'_{xy} n'_3,$$

$$L'_{22} = (\partial_y \mathbf{e}'_2)^T \mathbf{n}' = x'_{yy} n'_1 + y'_{yy} n'_2 + z'_{yy} n'_3. \quad (25)$$

We now have all the necessary quantities in order to compute H, K, k_1, k_2, ρ_1, ρ_2, $\mathbf{t}_1$, $\mathbf{t}_2$ for epoch t as well as the corresponding quantities H', K', k'_1, k'_2, ρ'_1, ρ'_2, $\mathbf{t}'_1$, $\mathbf{t}'_2$ for epoch t'. We need to compare the normal curvature at a direction (tangent unit vector) $\mathbf{t} = \mathbf{t}(\theta) = \cos\theta\,\mathbf{t}_1 + \sin\theta\,\mathbf{t}_2$, identified by its angle θ that forms with the principal direction $\mathbf{t}_1$. Using the above value (16) for $\mathbf{t}_1$ and $\mathbf{t}_2$ it is easy to verify that the component $\dot{\mathbf{u}} = \dot{\mathbf{u}}(\theta)$ of $\mathbf{t} = [\mathbf{e}_1 \mathbf{e}_2]\dot{\mathbf{u}} = [\mathbf{t}_1 \mathbf{t}_2] \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = [\mathbf{e}_1 \mathbf{e}_2] \begin{bmatrix} \mu_1 \rho_2 & \mu_2 \rho_2 \\ \mu_1 & \mu_2 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ are given by

$$\dot{\mathbf{u}} = \begin{bmatrix} \mu_1 \rho_1 & \mu_2 \rho_2 \\ \mu_1 & \mu_2 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \mu_1 \rho_1 \cos\theta + \mu_2 \rho_2 \sin\theta \\ \mu_1 \cos\theta + \mu_2 \sin\theta \end{bmatrix}, \quad (26)$$

In this case the curvature of the normal section at the direction $\mathbf{t} = \mathbf{t}(\theta)$ is given by the simple relation

$$k_\theta = k(\theta) = k_1 \cos\theta + k_2 \sin\theta\,. \quad (27)$$

To find the corresponding direction at epoch t', we note that the normal section with equation $\mathbf{x}(s)$ and tangent vector $\mathbf{t} = d\mathbf{x}/ds = \dot{\mathbf{x}} = (\partial_{\mathbf{u}}\mathbf{x})\dot{\mathbf{u}} = [\mathbf{e}_1 \mathbf{e}_2]\dot{\mathbf{u}}$, is deformed at epoch t' into a curve $\mathbf{x}'(s) = \mathbf{f}(\mathbf{x}(s))$, with tangent vector $\mathbf{t}' = d\mathbf{x}/ds = \dot{\mathbf{x}}' = (\partial_{\mathbf{u}'}\mathbf{x}')\dot{\mathbf{u}}' = [\mathbf{e}'_1 \mathbf{e}'_2]\dot{\mathbf{u}}'$. In our special case of convected coordinates $\mathbf{u}' = \mathbf{u}$, $\mathbf{t}' = [\mathbf{e}'_1 \mathbf{e}'_2]\dot{\mathbf{u}}$ and the angle $\theta' = \theta'(\theta)$ between $\mathbf{t}'$ and the new principal direction $\mathbf{t}'_1 = \mu'_1(\rho'_1 \mathbf{e}'_1 + \mathbf{e}'_2) = [\mathbf{e}'_1 \ \mathbf{e}'_2]\begin{bmatrix} \mu'_1 \rho'_1 \\ \mu'_1 \end{bmatrix}$ is given by

$$\cos\theta' = \mathbf{t}'^T \mathbf{t}'_1 = \dot{\mathbf{u}}^T \mathbf{G}' \begin{bmatrix} \mu'_1 \rho'_1 \\ \mu'_1 \end{bmatrix} = \begin{bmatrix} \mu_1 \rho_1 \cos\theta + \mu_2 \rho_2 \sin\theta \\ \mu_1 \cos\theta + \mu_2 \sin\theta \end{bmatrix}^T \mathbf{G}' \begin{bmatrix} \mu'_1 \rho'_1 \\ \mu'_1 \end{bmatrix} = A\cos\theta + B\sin\theta \,, \quad (28)$$

where

$$A = \mu_1 \rho_1 (g'_{11} \mu'_1 \rho'_1 + g'_{12} \mu'_1) + \mu_1 (g'_{12} \mu'_1 \rho'_1 + g'_{22} \mu'_1) \,,$$

$$B = \mu_2 \rho_2 (g'_{11} \mu'_1 \rho'_1 + g'_{12} \mu'_1) + \mu_2 (g'_{12} \mu'_1 \rho'_1 + g'_{22} \mu'_1) \,, \quad (29)$$

Differentiation of (28) gives

$$\frac{\partial \theta'}{\partial \theta} = \frac{A\sin\theta - B\cos\theta}{\sin\theta'} \,. \quad (30)$$

Note that the deformed curve $\mathbf{x}'(s)$ is not any more a normal section of the deformed surface, but we find it convenient to compare the curvature of the normal section $\mathbf{x}(s)$, not with the curvature of its image but rather with the curvature of the normal section having the same tangent vector as the image. Thus k_θ will be compared with the corresponding normal section curvature

$$k'_\theta = k'(\theta) = k'_1 \cos\theta'(\theta) + k'_2 \sin\theta'(\theta) \,. \quad (31)$$

The difference in curvature $\Delta k(\theta) = k'(\theta) - k(\theta)$, will take its extreme value at some particular direction $\hat{\theta}$, for which $\partial \Delta k / \partial \theta = 0$. Obviously

$$\frac{\partial \Delta k}{\partial \theta} = \frac{\partial k'}{\partial \theta'} \frac{\partial \theta'}{\partial \theta} - \frac{\partial k}{\partial \theta} = [-k'_1 \sin\theta' + k'_2 \cos\theta'] \frac{\partial \theta'}{\partial \theta} - [-k_1 \sin\theta + k_2 \cos\theta] =$$

$$= [-k'_1 \sin\theta' + k'_2 \cos\theta'] \frac{A\sin\theta - B\cos\theta}{\sin\theta'} - [-k_1 \sin\theta + k_2 \cos\theta] = 0 \quad (32)$$

or

$$-k'_1 + k'_2 \cot\theta' + \frac{k_1 - k_2 \cot\theta}{A - B\cot\theta} = 0 \,. \quad (33)$$

Noting that

$$\cot\theta' = \frac{\cos\theta'}{\sqrt{1 - \cos^2\theta'}} = \frac{A\cos\theta + B\sin\theta}{\sqrt{1 - [A\cos\theta + B\sin\theta]^2}} = \frac{A\cot\theta + B}{\sqrt{\cot^2\theta + 1 - (A\cot\theta + B)^2}} \quad (34)$$

and setting $X = \cot\theta$, we need to solve by numerical techniques the non-linear equation

$$\frac{-k_1+k_2X}{A-BX}+k_1'-k_2'\frac{AX+B}{\sqrt{1+X^2-(AX+B)^2}}=0. \qquad (35)$$

Using X we determine $\hat{\theta}$ and $\hat{\theta}'$ from $\cot\hat{\theta}=X$ and $\cot\hat{\theta}'=\dfrac{AX+B}{\sqrt{X^2+q-(AX+B)^2}}$, as well as the most differing curvatures

$$\hat{k}=k_1\cos\hat{\theta}+k_2\sin\hat{\theta}=\frac{1}{\hat{R}}, \qquad \hat{k}'=k_1'\cos\hat{\theta}'+k_2'\sin\hat{\theta}'=\frac{1}{\hat{R}'} \qquad (36)$$

Of more direct interest to the construction engineer are the original and final radii of curvature $\hat{R}$ and $\hat{R}'$, at epochs t and t' respectively, of the normal sections in conjugate material directions, with original direction $\hat{\theta}$ (measured from the principal axis $\mathbf{t}_1$ counter-clock-wise) corresponding to the pair of conjugate directions for which the maximum variation of normal curvature and thus the maximum "bending" of the surface appears. Particular cases are the developable surfaces constructed by bending a planar surface without local deformation. These have zero curvature k_2 at the direction of a line contained in the surface and we need only to compare the principle curvatures k_1 and k_1', for which $\Delta k=k'-k$ is maximized.

4. Interpolating heights and displacements

In order to compute the derivatives $z_x=\dfrac{\partial z}{\partial x}$, $z_y=\dfrac{\partial z}{\partial y}$, $z_{xx}=\dfrac{\partial^2 z}{\partial x^2}$, $z_{xy}=\dfrac{\partial^2 z}{\partial x\partial y}$, $z_{yy}=\dfrac{\partial^2 z}{\partial y^2}$, $x_x'=\dfrac{\partial x'}{\partial x}$, $x_y'=\dfrac{\partial x'}{\partial y}$, $y_x'=\dfrac{\partial y'}{\partial x}$, $y_y'=\dfrac{\partial y'}{\partial y}$, $z_x'=\dfrac{\partial z'}{\partial x}$, $z_y'=\dfrac{\partial z'}{\partial y}$, $x_{xx}'=\dfrac{\partial^2 x'}{\partial x^2}$, $x_{xy}'=\dfrac{\partial^2 x'}{\partial x\partial y}$, $x_{yy}'=\dfrac{\partial^2 x'}{\partial y^2}$, $y_{xx}'=\dfrac{\partial^2 y'}{\partial x^2}$, $y_{xy}'=\dfrac{\partial^2 y'}{\partial x\partial y}$, $y_{yy}'=\dfrac{\partial^2 y'}{\partial y^2}$, $z_{xx}'=\dfrac{\partial^2 z'}{\partial x^2}$, $z_{xy}'=\dfrac{\partial^2 z'}{\partial x\partial y}$, $z_{yy}'=\dfrac{\partial^2 z'}{\partial y^2}$ appearing in the above calculations, we must "construct" by interpolation the "height function" $z(x,y)$ and the "coordinate variation functions" $x'(x,y)$, $y'(x,y)$, $z'(x,y)$. Instead of the last three it is usually convenient to set $x'=x+\Delta x$, $y'=y+\Delta y$, $z'=z+\Delta z$ and seek rather the displacement functions $\Delta x(x,y)$, $\Delta y(x,y)$, $\Delta z(x,y)$. Thus we need to interpolate for four functions on the basis of the available discrete data $z_i=z(x_i,y_i)$, $\Delta x_i=\Delta x(x_i,y_i)$, $\Delta y_i=\Delta y(x_i,y_i)$, $\Delta z_i=\Delta z(x_i,y_i)$, $i=1,2,...,n$. In general a function $f(P)=f(x,y)$ with discrete data $f_i=f(P_i)=f(x_i,y_i)$ can be interpolated by modelling as a linear combination of known basis functions of the form $f(P)=a_1\varphi_1(P)+a_2\varphi_2(P)+...+a_m\varphi_m(P)$. For all the data the corresponding set of n equations $f_i=a_1\varphi_1(P_i)+a_2\varphi_2(P_i)+...+a_m\varphi_m(P_i)$, $i=1,...,n$ in the m unknowns a_k, $k=1,...,m$, takes the matrix form $\mathbf{f}=\mathbf{Fa}$, with $F_{ik}=\varphi_k(P_i)$. When $n>m$ there is no exact solution but instead we may set $\mathbf{f}=\mathbf{Fa}+\mathbf{v}$ and obtain a least-squares smoothing interpolation through $\hat{\mathbf{a}}=(\mathbf{F}^T\mathbf{PF})^{-1}\mathbf{F}^T\mathbf{Pf}$, for a positive-definite weight matrix $\mathbf{P}$. For $n=m$ and exact interpolation is obtained through $\hat{\mathbf{a}}=\mathbf{F}^{-1}\mathbf{f}$. However the most flexible case is that of exact interpolation with $n < m$ and an infinite number of solutions, in which case a unique one $\hat{\mathbf{a}} = \mathbf{W}^{-1}\mathbf{F}^T(\mathbf{F}\mathbf{W}^{-1}\mathbf{F}^T)^{-1}\mathbf{f}$, for some preferably diagonal weight matrix $\mathbf{W}$. Setting $(\mathbf{f}_P)_k = \varphi_k(P)$, the interpolated function takes the form $f(P) = \sum_{k=1}^{m} \hat{a}_k \varphi_k(P) = \mathbf{f}_P^T \hat{\mathbf{a}} =$ $= \mathbf{f}_P^T \mathbf{W}^{-1}\mathbf{F}^T(\mathbf{F}\mathbf{W}^{-1}\mathbf{F}^T)^{-1}\mathbf{f}$. If the two point function $k(P,Q) = \sum_{k=1}^{m} W_{kk}^{-1} \varphi_k(P)\varphi_k(Q)$ is introduced, the solution can be also written in the compact form $f(P) = \mathbf{k}_P^T \mathbf{K}^{-1}\mathbf{f}$, where $K_{ij} = k(P_i, P_j)$ and $(\mathbf{k}_P)_i = k(P, P_i)$. Thus the solution can be obtained even without explicitly defining the weights W_{kk} or the base functions $\{\varphi_k(P)\}$, by introducing rather directly the function $k(P,Q)$. A probabilistic interpretation is possible if we set $W_{kk} = 1/\sigma_k^2$ and interpret σ_k^2 as the variances of the zero-mean uncorrelated coefficients a_k. When a_k are considered random variables $f(P) = \sum_{k=1}^{m} a_k \varphi_k(P)$ becomes a zero-mean stochastic process (random function) and $k(P,Q) = E\{f(P)f(Q)\}$ is simply the covariance function of $C(P,Q)$ of $f(P)$. The interpolating equations $f(P) = \mathbf{k}_P^T \mathbf{K}^{-1}\mathbf{f}$ become in this case the (minimum mean square error) prediction equations

$$f(P) = \mathbf{c}_P^T \mathbf{C}^{-1}\mathbf{f} \tag{37}$$

well known in geodesy under the jargon name “collocation”, with $C_{ij} = C(P_i, P_j)$ and $(\mathbf{c}_P)_i = C(P, P_i)$, determined from the single choice of a positive-definite covariance function $C(P,Q) = C(x_P, y_P; x_Q, y_Q)$. Since the prediction point P appears only in the vector $\mathbf{c}_P$, it is possible to obtain the required first and second order partial derivatives of $f(P) = f(x,y)$ by differentiating directly the elements $(\mathbf{c}_P)_i = C(P, P_i) = C(x, y; x_i, y_i)$. For example $\frac{\partial^2 f}{\partial x \partial y}(x,y) = \mathbf{c}^T \mathbf{C}^{-1}\mathbf{f}$ with $c_i = \frac{\partial^2 f}{\partial x \partial y}(\mathbf{c}_P)_i = \frac{\partial^2 f}{\partial x \partial y} C(x, y; x_i, y_i)$ and similar relations hold for the other partial derivatives. Replacing $f(x,y)$ with $z(x,y)$, $\Delta x(x,y)$, $\Delta y(x,y)$, $\Delta z(x,y)$, we may interpolate-predict the required derivatives, e.g. $x'_x = \frac{\partial x'}{\partial x}$, $\frac{\partial^2 \Delta x}{\partial x \partial y}(x,y)$, etc., and hence the required ones, e.g. $\frac{\partial x'}{\partial x} = \frac{\partial (x + \Delta x)}{\partial x} = 1 + \frac{\partial \Delta x}{\partial x}$, $\frac{\partial^2 x'}{\partial x \partial y} = \frac{\partial^2 (x + \Delta x)}{\partial x \partial y} = \frac{\partial^2 \Delta x}{\partial x \partial y}$, etc. With all the derivatives at hand we proceed to the computation of the matrices $\mathbf{G}$, $\mathbf{L}$, $\mathbf{G}'$, $\mathbf{L}'$, using equations (8), (9), (24) and (25), which constitute the basis for the determination of the principal normal curvatures k_1, k_2, k'_1, k'_2, using (14) and (13). Solving equation (35) we determine the angles of maximum bending $\hat{\theta}$, $\hat{\theta}'$ and finally (using equation 36) the corresponding curvatures $\hat{k}$, $\hat{k}'$ and the radii of curvature $\hat{R} = 1/\hat{k}$, $\hat{R}' = 1/\hat{k}'$.

Before interpolating (in particular when coordinates $\mathbf{x}_i$ and $\mathbf{x}'_i$ do not refer to the same reference system), it is necessary to perform a “trend removal” by a least-squares fitting where the coordinates $\mathbf{x}'_i$ are rotated ($\mathbf{R}$) and translated ($\mathbf{d}$) into a new set $\tilde{\mathbf{x}}_i = \mathbf{R}\mathbf{x}'_i + \mathbf{d}$, satisfying $\sum_i |\mathbf{x}_i - \tilde{\mathbf{x}}_i|^2 = \min$.

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