Time series analysis of 3D coordinates using nonstochastic observations

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Abstract. Adjustment and testing of a combination of stochastic and nonstochastic observations is applied to the deformation analysis of a time series of 3D coordinates. Nonstochastic observations are constant values that are treated as if they were observations. They are used to formulate constraints on the unknown parameters of the adjustment problem. Thus they describe deformation patterns. If deformation is absent, the epochs of the time series are supposed to be related via affine, similarity or congruence transformations. S-basis invariant testing of deformation patterns is treated. The model is experimentally validated by showing the procedure for a point set of 3D coordinates, determined from total station measurements during five epochs. The modelling of two patterns, the movement of just one point in several epochs, and of several points, is shown. Full, rank deficient covariance matrices of the 3D coordinates, resulting from free network adjustments of the total station measurements of each epoch, are used in the analysis.

Keywords. Geodetic deformation analysis, Time series, 3D coordinates, Full, singular covariance matrices, Nonstochastic observations, S-basis invariant testing.

1 Introduction

Geodetic deformation analysis is about change of form and size of the earth's surface or of objects on, below or above it, and also of the relative position and orientation of the objects. The objects to be analysed are represented by points that constitute a three-dimensional geodetic network. It is nowadays common practice to use total stations, GPS receivers and other devices for the analysis. If processing the measurements results in three-dimensional x, y, z coordinates, these can be presented in two-dimensional graphs, showing the displacements in time or in space or both. It is, however, generally difficult to come to statistically sound conclusions by analysing the graphs. Computational methods to test statistical hypotheses are desirable. For two epochs methods are available to perform an adjustment of the coordinates, taking into account their covariance matrix, and to perform hypothesis testing (Heunecke et al, 2013, p. 494ff.). The analysis is generally not invariant for the used S-basis (Velsink, 2015c).

In this paper an adjustment model is proposed that analyses a time series of 3D coordinates, taking account of the covariance matrices and analysing the deformations of all points and all epochs simultaneously, by computing statistics of deformation patterns and testing them. The model can be applied to any 3D geodetic network, observed quasi-continuously (i.e. with permanently installed sensors measuring frequently). Examples are the monitoring of the movement of a subset of points through all epochs, or the periodic oscillation of a subset of points.

In the next section the problem is defined. After describing existing approaches in section 3, section 4 describes the solution set-up. The adjustment model is treated in section 5. The adjustment itself, the deformation testing and the S-basis invariance are handled in section 6. Section 7 gives an experimental validation of the model.

2 Problem definition

The problem addressed in this paper is the adjustment and testing for deformations of a time series of three-dimensional coordinates of a geodetic network, with a covariance matrix of the coordinates that is full and generally singular, because each epoch of the time series is adjusted as a free network.

The described problem will be handled by constructing a least squares adjustment model. As a

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practical application to show the usability of the model, the continuous monitoring by a total station of points, situated on built structures that are prone to deformations, is analysed.

3 Existing solutions

Heunecke et al. give an overview of existing approaches for deformation analysis of two time epochs of deformation measurements (Heunecke et al, 2013, p. 521) . The general approach is to compute displacement vectors between coordinates of two epochs and their covariance matrix. Different approaches exists to analyse the displacement vectors, e.g. by using 95%-confidence ellipsoids after a least squares adjustment (Kamiński and Nowel, 2013; Caspary, 2000) or a L1-norm adjustment (Chen, 1983; Caspary and Borutta, 1987), or using constraints on common points and analysing the quadratic form of the weighted estimated least squares residuals (Heunecke et al, 2013, p. 500ff.).

Heunecke et al. give also methods to analyse time series (Heunecke et al, 2013, p. 548) . They do not take advantage of the covariance matrices of the coordinates and do not perform the analysis for all three dimensions (x, y and z) simultaneously. As a consequence the choice of datum definition and the solution's invariance for it, are not addressed.

A comprehensive 3D multi-epoch model is treated by Caspary (2000, p. 164ff.). It takes care of singular covariance matrices and incorporates deterministic deformation models. It assumes all epochs to be defined relative to the same S-basis, which has to be defined by points, measured in all epochs. Testing is treated for the sequential adjustment case. Quality description of the tests is not treated.

4 Solution set-up

4.1 Form and size, position and orientation

The subject of geodetic deformation analysis is the change in time of the form and size of objects, and also of the relative position and orientation of the objects. Form, size, relative position and orientation can be recorded by Euclidian x, y, z coordinates. It is assumed that the coordinates are normally distributed with a probability density function, which is fully described by a known covariance matrix, except for the first moments. This matrix may be singular, e.g. because it stems from a free network adjustment. If there are reference points, i.e. points that are considered not to be influenced by the deformation to be analysed, they are part of the geodetic network, and are analysed simultaneously with the object points.

The Euclidian coordinates describe the position and orientation of the network relative to the coordinate origin and axes as well. These, however, are *not* subject of the analysis. Their uncertaintity, as it is reflected in the covariance matrix, has to be eliminated from the analysis. This is realised in the adjustment model by a congruence or similarity transformation of the coordinates of each epoch to the coordinate system of the reference epoch. It is shown that after these transformations, testing of deformation hypotheses can be done independently from the S-bases chosen for the individual epochs. The first epoch is chosen in this paper as reference epoch, but any other epoch as reference epoch would give the same analysis results.

The choice between a congruence and a similarity transformation depends on the question, whether the scale (unit of length) is considered stable between epochs and essential for the analysis.

The set-up of the adjustment model, with transformations between the epochs incorporated into it, not only removes the influence of origin, axes and scale of the reference system on the analysis. It also makes it possible to test, without additional S-transformations, for deformations of all kinds of subsets of points, independent of their being reference or object points, or being part of the S-basis or not. It is possible to include in one hypothesis that is to be tested, both reference and object points, and both points within and outside the S-basis.

4.2 Nonstochastic observations

The adjustment model is built as a model of observation equations with the coordinates as observations, arranged according to the epochs. The parameters are the expectations of the coordinates of all epochs. Each point has for each epoch different coordinates in the parameter vector. Also the transformation parameters of each epoch relative to the previous one appear in the parameter vector.

Constraints are stated concerning the coordinates of all epochs. In the case of stability analysis the constraints state that the expectations of coordinates of the same points in different epochs are equal. These constraints are added to the observation vector as nonstochastic observations, following the approach of Velsink (2015b). If coordinates are assumed to be subject to some kind of deformation, for example a linear movement of one or more points, or a deformation pattern with a periodic character for a subset of points, the constraints add extra unknown parameters, for example the linear rate of movement, or the coefficients of the periodic pattern, to the parameter vector.

The advantage of using nonstochastic observations is that testing of deformation hypotheses is done in the same way as testing of one- or multi-dimensional hypotheses on biases in the other observations. Least squares estimates of the deformations are determined using standard formulas. Also minimal detectable biases can be computed with standard formulas, giving information on the deformation sizes that can be detected with the tests.

4.3 Full, singular covariance matrices

Observations, for example direction and distance observations of total stations, and their stochastic model are used for a deformation analysis, which is performed in two phases. In the first phase the direction and distance observations are adjusted for each epoch separately. The results are coordinates and their covariance matrices for all epochs. The second step is the subject of this paper: the deformation analysis of the coordinates of many epochs. The covariance matrices of all epochs have to be used (Tienstra, 1956, p. 154). These matrices are generally full matrices (no or few zeros) and singular, because each epoch is adjusted as a free network, not connected to control points. The adjustment model of section 5 can handle full, singular covariance matrices.

4.4 Solution characteristics

An overview of the solution characteristics can now be given. The most relevant terms are listed below and the solution procedure is illustrated by a Nassi-Schneidermann diagram (figure 1).

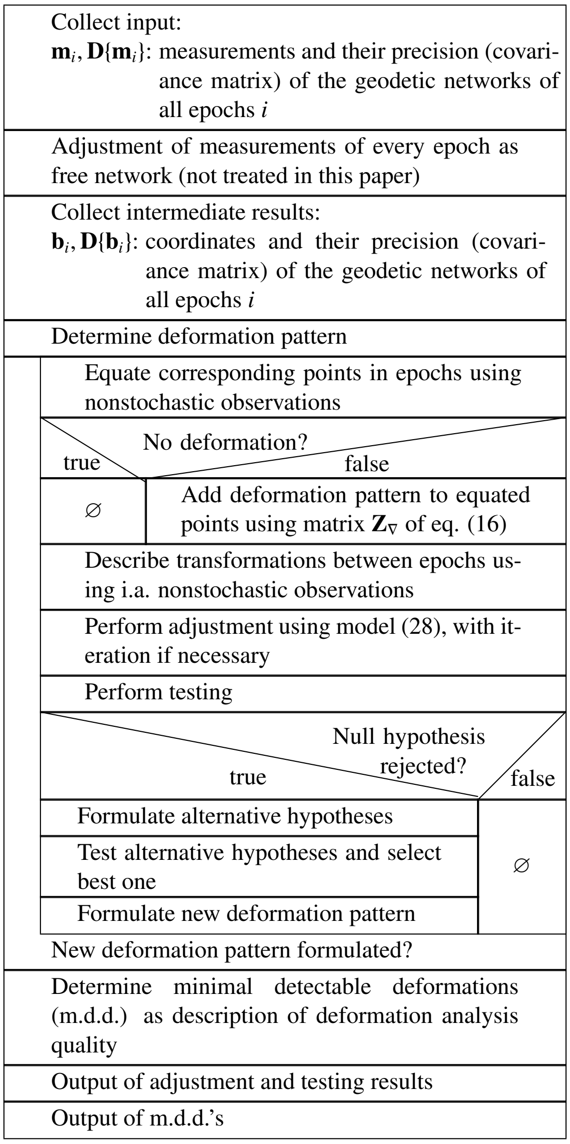

Fig. 1: Solution procedure

A geodetic network per epoch is a set of points on, above, or under the earth's surface, in this paper assumed to be represented by 3D Euclidian coordinates.

Form and size of a geodetic network (and their changes in time) are of interest, not the position and orientation relative to the reference system. Transformations are therefore included in the adjustment model.

Stability assumes the expectations of coordinates to be equal through all epochs, except for the above mentioned transformations.

A deformation pattern is the relation between the geodetic networks per epoch, formulated by giving the expectations of coordinates through the epochs using mathematical functions, described by nonstochastic observations, which depend on unknown deformation parameters, for example a linear movement rate, or the coefficients of a series expansion of a periodic function.

Inside or outside the adjustment model we put the description of the deformation. If it is *inside*, nonstochastic observations describe the deformation pattern, and extra deformation parameters are included in the parameter vector. If it is *outside*, the adjustment model assumes *stability* and no extra deformation parameters are in the parameter vector. The hypothesis of stability is tested against alternative hypotheses, describing deformation patterns, by appropiate test statistics, which make use of the nonstochastic observations to determine matrix $\mathbf{Z}_\nabla$ of eq. (16).

Singular, full covariance matrices result from the free network adjustments of each epoch, and are used in the adjustment model. If only coordinates are available for each epoch, a substitute matrix, for example a unit matrix, is used as covariance matrix, which yields sub optimal adjustment and testing results.

5 Adjustment model

5.1 Observations and parameters

The adjustment model is built taking as:

- observations:
  1. cartesian 3D point coordinates of a geodetic network and their covariance matrix, available for at least two epochs. For the first epoch they are assembled in vector $\underline{\mathbf{a}}_1$ (an underlined variable indicates a stochastic variable) with the covariance matrix $\mathbf{D}\{\underline{\mathbf{a}}_1\}$. For the second and later epochs they are assembled in vectors $\underline{\mathbf{b}}_i$, with i the epoch number, which runs from 2 to p with p the number of epochs. Each $\underline{\mathbf{b}}_i$ has a covariance matrix $\mathbf{D}\{\underline{\mathbf{b}}_i\}$;
  2. nonstochastic observations $\mathbf{z}_f$, describing constraints on the transformation parameters; their covariance matrix is the zero matrix;
  3. nonstochastic observations $\mathbf{z}_d$, describing the deformation pattern; their covariance matrix is the zero matrix.
- unknown parameters:
  1. expectations of cartesian 3D network point coordinates, for each epoch assembled in vector $\mathbf{c}_i$ of epoch i. Vector $\mathbf{c}$ takes all epochs together:
$$\mathbf{c} = \left(\mathbf{c}_1, \cdots, \mathbf{c}_p\right)^T. \tag{1}$$
  2. vector of transformation parameters $\mathbf{f}$, subdivided in subvectors $\mathbf{f}_{i,i-1}$ for the transformation in each epoch interval between epoch i and $i-1$, with $i = 2, \ldots, p$.
  3. additonal parameters ∇ to describe the trend function of the deformation, see section 4.2.

5.2 Nonlinear adjustment model

In the adjustment model the expectations of all point coordinates are expressed in the reference system of the first epoch, and are *parameters* in vector $\mathbf{c}$. The *observed* coordinates in the first epoch are taken together in vector $\underline{\mathbf{a}}_1$. We have:

$$E\{\underline{\mathbf{a}}_1\} = \mathbf{c}_1 = \mathbf{P}_1\,\mathbf{c}, \tag{2}$$

with $E\{.\}$ the expectation operator, and $\mathbf{P}_1$ the matrix that selects the points of the first epoch from $\mathbf{c}$. $\mathbf{P}_1$ has only ones and zeros. The observed coordinates $\underline{\mathbf{b}}_i$ in a following epoch i ($i = 2, \cdots, p$) are assumed to be in a separate reference system, indicated by a superindex (i):

$$\underline{\mathbf{b}}_i = \underline{\mathbf{b}}_i^{(i)}. \tag{3}$$

These coordinates are transformed with a vector function $\varphi_{i,i-1}$ to the reference system of epoch $(i-1)$:

$$\underline{\mathbf{b}}_i^{(i-1)} = \varphi_{i,i-1}(\underline{\mathbf{b}}_i^{(i)}, \mathbf{f}_{i,i-1}), \tag{4}$$

then with $\varphi_{i-1,i-2}$ to the reference system of epoch $(i-2)$, and so on, and we get the transformed coordinates $\underline{\mathbf{a}}_i$:

$$\underline{\mathbf{a}}_i = \underline{\mathbf{b}}_i^{(1)} = \varphi_{2,1}(\cdots(\varphi_{i,i-1}(\underline{\mathbf{b}}_i, \mathbf{f}_{i,i-1}), ...), \mathbf{f}_{2,1}), \tag{5}$$

and:

$$E\{\underline{\mathbf{a}}_i\} = \mathbf{c}_i = \mathbf{P}_i\,\mathbf{c}. \tag{6}$$

$\mathbf{P}_i$ selects $\mathbf{c}_i$ from $\mathbf{c}$. It follows that:

$$E\{\varphi_{2,1}(\cdots(\varphi_{i,i-1}(\underline{\mathbf{b}}_i, \mathbf{f}_{i,i-1}), ...), \mathbf{f}_{2,1})\} = \mathbf{P}_i\,\mathbf{c}. \tag{7}$$

Hopping from epoch i through all intermediate epochs to the first one, is chosen, and not a direct transformation to the first epoch, because it is assumed that in general more common points are available for successive epochs.

Following the approach of Velsink (2015a), the transformation $\varphi_{i,j}$ between epoch i and j, is of a general form, for example an affine transformation, which is changed to another type of transformation, for example a similarity transformation, by the use of constraints. These constraints are formulated as nonstochastic observations:

$$\mathbf{z}_f = \zeta_f(\mathbf{f}), \tag{8}$$

for which zeros are assumed as observed values.

The deformation pattern is described by a vector of nonstochastic observations $\mathbf{z}_d$ and a deformation function ζ_d, which gives a relation between the elements of $\mathbf{c}$ and the elements of a vector of deformation parameters ∇:

$$\mathbf{z}_d = \zeta_d(\mathbf{c}, \nabla). \tag{9}$$

For $\mathbf{z}_d$ we assume zeros as the observed values.

From equations (2), (7), (8) and (9) follows the following system for p epochs:

$$\begin{cases} E\{\underline{\mathbf{a}}_1\} & = \mathbf{P}_1\mathbf{c}, \\ E\{\varphi_{2,1}(\underline{\mathbf{b}}_2, \mathbf{f}_{2,1})\} & = \mathbf{P}_2\mathbf{c}, \\ & \vdots \\ E\{g(\underline{\mathbf{b}}_p, \mathbf{f})\} & = \mathbf{P}_p\mathbf{c}, \\ \mathbf{z}_f & = \zeta_f(\mathbf{f}), \\ \mathbf{z}_d & = \zeta_d(\mathbf{c}, \nabla), \end{cases} \tag{10}$$

with

$$g(\underline{\mathbf{b}}_p, \mathbf{f}) = \varphi_{2,1}(...(\varphi_{p,p-1}(\underline{\mathbf{b}}_p, \mathbf{f}_{p,p-1}), ...), \mathbf{f}_{2,1}).$$

A point may be present in an epoch, but missing in one or more other epochs. This is handled by matrix $\mathbf{P}_i$. The S-basis definition of an epoch is arbitrary (see section 6.3) and may be realised by only a few points, by many points, or by all. There can even be no S-basis, i.e. the covariance matrix is regular, and the S-basis can be considered to lie outside the geodetic network. The fact that a point is missing, be it in the first or in any other epoch, does therefore not pose any problem for the deformation analysis with model (10).

5.3 Transformations

5.3.1 Affine Transformation

As general form of transformation $\varphi_{i,j}$ the affine transformation is taken, written as:

$$\begin{aligned} &\begin{pmatrix} \mathbf{x}^T \\ \mathbf{y}^T \\ \mathbf{z}^T \end{pmatrix} = \mathbf{R} \begin{pmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{pmatrix} + \mathbf{t}\boldsymbol{\epsilon}, \\ &\mathbf{R} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ \mathbf{t} = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}, \\ &\boldsymbol{\epsilon} = \begin{pmatrix} 1, 1, \ldots, 1 \end{pmatrix}. \end{aligned} \tag{11}$$

The column vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ contain resp. the x-, y-, z-coordinates of $\mathbf{b}_i$ before transformation. The vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ contain the coordinates after transformation. $\mathbf{R}$ describes the rotation, shear and scale change of the affine transformation, $\mathbf{t}$ the translation.

5.3.2 Congruence Transformation

Equation (11) describes a congruence (or rigid body) transformation, if the nine coefficients of matrix $\mathbf{R}$ meet the following six constraints:

$$\begin{aligned} &\mathbf{a}_i^T\mathbf{a}_j = \delta_{ij}, \ \mathbf{a}_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{pmatrix}, \ i,j = 1,2,3, \\ &j \geq i, \ \delta_{ij} = 1 \text{ if } i = j, \text{ otherwise } \delta_{ij} = 0. \end{aligned} \tag{12}$$

In the following sections a linearised adjustment model is derived. The linearised constraints are:

$$\begin{pmatrix} \mathbf{a}_2^{0T} & \mathbf{a}_1^{0T} & \mathbf{0} \\ \mathbf{a}_3^{0T} & \mathbf{0} & \mathbf{a}_1^{0T} \\ \mathbf{0} & \mathbf{a}_3^{0T} & \mathbf{a}_2^{0T} \\ \mathbf{a}_1^{0T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_2^{0T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{a}_3^{0T} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{a}_1 \\ \Delta\mathbf{a}_2 \\ \Delta\mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{13}$$

where $\mathbf{0}$ is the (1×3) zero vector and $\mathbf{a}_i^0$ (i=1,2,3) is the vector of approximate values of $\mathbf{a}_i$. Δ indicates the difference of the quantity concerned and its approximate value.

5.3.3 Similarity transformation

For the similarity transformation the affine transformation is constrained with five constraints. Three constraints state that the three rows of $\mathbf{R}$ are perpendicular to each other. The two remaining constraints state that the lengths of the first and second row, and those of the second and third row are equal. The linearised constraints are:

$$\begin{pmatrix} \mathbf{a}_2^{0^T} & \mathbf{a}_1^{0^T} & \mathbf{0} \\ \mathbf{a}_3^{0^T} & \mathbf{0} & \mathbf{a}_1^{0^T} \\ \mathbf{0} & \mathbf{a}_3^{0^T} & \mathbf{a}_2^{0^T} \\ \mathbf{a}_1^{0^T} & -\mathbf{a}_2^{0^T} & \mathbf{0} \\ \mathbf{a}_1^{0^T} & \mathbf{0} & -\mathbf{a}_3^{0^T} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{a}_1 \\ \Delta\mathbf{a}_2 \\ \Delta\mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (14)$$

5.3.4 Approximate transformation

Before the adjustment, $\underline{\mathbf{b}}_i$ is approximately transformed to $\underline{\mathbf{b}}_i'$ in the reference system of $\underline{\mathbf{a}}_1$, using equation (5). Likewise $\mathbf{D}\{\underline{\mathbf{b}}_i\}$ is transformed to $\mathbf{D}\{\underline{\mathbf{b}}_i'\}$ by applying the law of propagation of covariances. The approximate transformation parameters are determined as affine parameters, and subsequently adapted to those of a congruence or similarity transformation using singular value decomposition (Velsink, 2015a). The transformations of equation (10) are now differential transformations. In each iteration step of the adjustment this is repeated with adapted transformation parameters from the previous iteration step. Therefore as approximate values for a_{ij} in the constraints of the congruence or similarity transformation we can take $a_{ij}^0 = \delta_{ij}$.

In the following sections, if $\underline{\mathbf{b}}_i$ is written, $\underline{\mathbf{b}}_i'$ is meant.

5.4 Linearised adjustment model

A linearised adjustment model is built for the deformation analysis. Linearisation of all equations of system (10) is done with implicit differentiation relative to the observed vectors $\underline{\mathbf{a}}_1$, $\underline{\mathbf{b}}_i$ (i= 2,$\cdots$,p), $\mathbf{z}_d$, and $\mathbf{z}_f$, and the unknown parameter vectors $\mathbf{f}$, $\mathbf{c}$ and ∇.

The first and last two equations of system (10) are linearised as:

$$\begin{cases} E\{\underline{\Delta\mathbf{a}}_1\} &= \mathbf{P}_1\,\Delta\mathbf{c}, \\ \Delta\mathbf{z}_f &= (\dfrac{\partial\zeta_f}{\partial\mathbf{f}})_0\Delta\mathbf{f}, \\ \Delta\mathbf{z}_d &= (\dfrac{\partial\zeta_d}{\partial\mathbf{c}})_0\Delta\mathbf{c} + (\dfrac{\partial\zeta_d}{\partial\nabla})_0\Delta\nabla. \end{cases} \qquad (15)$$

We define for later use:

$$\begin{aligned} \mathbf{Z}_f &= (\frac{\partial\zeta_f}{\partial\mathbf{f}})_0, \\ \mathbf{Z}_d &= (\frac{\partial\zeta_d}{\partial\mathbf{c}})_0,\ \mathbf{Z}_\nabla = (\frac{\partial\zeta_d}{\partial\nabla})_0. \end{aligned} \qquad (16)$$

The partial derivatives of the vectors ζ_f and ζ_d with respect to the vectors $\mathbf{f}$, $\mathbf{c}$ and ∇ are matrices. The parentheses with zero $(.)_0$ indicate that approximate values of the parameters have to be used to get the values in the matrices.

For the equations with $\underline{\mathbf{b}}_i$ (i=2,$\cdots$,p) in system (10) the linearised equations are:

$$\mathbf{B}_i\,E\{\underline{\Delta\mathbf{b}}_i\} + \mathbf{F}_i\,\Delta\mathbf{f}_i = \mathbf{P}_i\,\Delta\mathbf{c}, \qquad (17)$$

with the matrices $\mathbf{B}_i$ defined as follows:

$$\mathbf{B}_i = \mathbf{B}_{2,1}\mathbf{B}_{3,2}\cdots\mathbf{B}_{i,i-1}, \qquad (18)$$

and with (j=2,...,$i-1$):

$$\begin{aligned} \mathbf{B}_{j,j-1} &= (\frac{\partial\varphi_{j,j-1}}{\partial\varphi_{j+1,j}})_0 = (\frac{\partial\varphi_{j,j-1}}{\partial\mathbf{b}_i^{(j)}})_0, \\ \mathbf{B}_{i,i-1} &= (\frac{\partial\varphi_{i,i-1}}{\partial\mathbf{b}_i^{(i)}})_0. \end{aligned} \qquad (19)$$

$\mathbf{F}_i$ is defined for $i = 2\ldots p$ as follows:

$$\mathbf{F}_i = \left(\mathbf{F}_{2,1},\cdots,\mathbf{F}_{i,i-1},\mathbf{0},\cdots,\mathbf{0}\right), \qquad (20)$$

with (p-i) matrices $\mathbf{0}$ of zeros, which have the same number of rows as $\mathbf{F}_{2,1}$, and the partioning of $\mathbf{F}_i$ in columns in accordance with the partitioning of $\Delta\mathbf{f}$:

$$\Delta\mathbf{f} = \left(\Delta\mathbf{f}_{2,1},\cdots,\Delta\mathbf{f}_{i,i-1},\Delta\mathbf{f}_{i+1,i}\cdots,\Delta\mathbf{f}_{p,p-1}\right). \qquad (21)$$

For $\mathbf{F}_{i,i-1}$ (i=2,...,p) we have:

$$\mathbf{F}_{i,i-1} = \mathbf{B}_{2,1}\,\mathbf{B}_{3,2}\cdots\mathbf{B}_{i-2,i-1}\,(\frac{\partial\varphi_{i,i-1}}{\partial\mathbf{f}_{i,i-1}})_0. \qquad (22)$$

Matrix $\mathbf{B}_{i,i-1}$ for an affine transformation is given by Velsink (2015a) as follows:

$$\mathbf{B}_{i,i-1} = \begin{pmatrix} a_{11}^0\mathbf{I} & a_{12}^0\mathbf{I} & a_{13}^0\mathbf{I} \\ a_{21}^0\mathbf{I} & a_{22}^0\mathbf{I} & a_{23}^0\mathbf{I} \\ a_{31}^0\mathbf{I} & a_{32}^0\mathbf{I} & a_{33}^0\mathbf{I} \end{pmatrix}, \qquad (23)$$

with a_{ij}^0 (i,j=1, 2, 3) the approximate values of a_{ij} and $\mathbf{I}$ the ($n \times n$) unit matrix and n the amount of points in $\underline{\mathbf{b}}_i$.

As explained in section 5.3.4, we can take $a_{ij}^0 = \delta_{ij}$, which results in a unit matrix for $\mathbf{B}_{i,i-1}$, from which follows, see equations (18) and (22):

$$\begin{aligned} \mathbf{B}_i &= \mathbf{I}, \\ \mathbf{F}_{i,i-1} &= (\frac{\partial \varphi_{i,i-1}}{\partial \mathbf{f}_{i,i-1}})_0. \end{aligned} \tag{24}$$

Matrix $\mathbf{F}_{i,i-1}$ for an affine transformation is given by Velsink (2015a) as follows:

$$\mathbf{F}_{i,i-1} = \begin{pmatrix} \boldsymbol{\beta}_i & \mathbf{0} & \mathbf{0} & \boldsymbol{\epsilon}_1 \\ \mathbf{0} & \boldsymbol{\beta}_i & \mathbf{0} & \boldsymbol{\epsilon}_2 \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\beta}_i & \boldsymbol{\epsilon}_3 \end{pmatrix}, \tag{25}$$

where $\boldsymbol{\beta}_i$, $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$, $\boldsymbol{\epsilon}_3$ and $\mathbf{0}$ are all $(n \times 3)$ matrices, as follows:

$\boldsymbol{\beta}_i = (\mathbf{u}_0, \mathbf{v}_0, \mathbf{w}_0)$; $\mathbf{u}_0$, $\mathbf{v}_0$, $\mathbf{w}_0$ are approximate values of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ (the x, y, z coordinates in $\underline{\mathbf{b}}_i$), which can be transformed to make the barycentre the origin.

$$\boldsymbol{\epsilon}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \end{pmatrix},$$

$\boldsymbol{\epsilon}_2$ and $\boldsymbol{\epsilon}_3$ are analogous matrices as $\boldsymbol{\epsilon}_1$ with ones in the second, resp. third column,

$\mathbf{0}$ is the $(n \times 3)$ zero matrix.

We define $\mathbf{F}_1$ as the null matrix $\mathbf{0}$ and put it together with the $\mathbf{F}_i$, $i = 2 \dots p$ of equation (20) into matrix $\mathbf{F}$. Analogously we take all $\mathbf{P}_i$ together in a matrix $\mathbf{P}$:

$$\begin{aligned} \mathbf{F} &= \left(\mathbf{F}_1, \dots, \mathbf{F}_p\right)^T \\ \mathbf{P} &= \left(\mathbf{P}_1, \dots, \mathbf{P}_p\right)^T \end{aligned} \tag{26}$$

We define vector $\Delta\mathbf{b}$ as:

$$\Delta\mathbf{b} = \left(\Delta\mathbf{a}_1, \Delta\mathbf{b}_2, \dots, \Delta\mathbf{b}_p\right)^T. \tag{27}$$

We can now formulate the linearised equivalent of system (10):

$$E\{\begin{pmatrix} \underline{\Delta\mathbf{b}} \\ \Delta\mathbf{z}_f \\ \Delta\mathbf{z}_d \end{pmatrix}\} = \begin{pmatrix} \mathbf{P} & -\mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_f & \mathbf{0} \\ \mathbf{Z}_d & \mathbf{0} & \mathbf{Z}_\nabla \end{pmatrix} \begin{pmatrix} \Delta\mathbf{c} \\ \Delta\mathbf{f} \\ \Delta\nabla \end{pmatrix}. \tag{28}$$

The covariance matrix of the observation vector on the left-hand side consists of the covariance matrices of $\mathbf{a}_1$ and $\mathbf{b}_i$, i=2, $\cdots$,p, as described in section 5.1, approximately transformed as described in section 5.3.4, and zero matrices for the remainder if no correlation between the epochs is assumed (which is, however, not necessary to solve the model).

The model takes each epoch as a separate geodetic network: each point has a different point number for each epoch, for example point A is called A_1 in epoch 1, A_2 in epoch 2, etc. The hypothesis that no deformation has occured is formulated by stating that

$$\left\{ \begin{aligned} 0 &= x_{A_2} - x_{A_1}, \\ 0 &= y_{A_2} - y_{A_1}, \\ 0 &= z_{A_2} - z_{A_1}, \\ &\text{etc.} \end{aligned} \right. \tag{29}$$

The separate geodetic networks are linked together in this way. Equation (29) gives the nonstochastic observation equations (the zeros constitute together vector $\mathbf{z}_d$ and have a standard deviation of zero). The number of rows of matrix $\mathbf{Z}_d$ is three times the number of points. In each row there are zeros and one 1 and one -1 for respectively the coordinate of epoch 2 and epoch 1 (which are separate unknowns in the parameter vector). There are no parameters ∇ and no matrix $\mathbf{Z}_\nabla$.

Let us now assume that a deformation is present for point A. Let it be a linear movement for which we write:

$$\begin{aligned} 0 &= x_{A_2} - x_{A_1} + a_x t_{12}, \\ 0 &= y_{A_2} - y_{A_1} + a_y t_{12}, \\ 0 &= z_{A_2} - z_{A_1} + a_z t_{12} \end{aligned} \tag{30}$$

The a_x, a_y, a_z are unknown parameters, which enter the parameter vector ∇, and for which a least squares estimate is determined in the adjustment. t_{12} is the time interval between epoch 1 and 2. The matrix $\mathbf{Z}_\nabla$ is in this case a matrix with three columns and three elements t_{12} on the rows of the three nonstochastic observations mentioned, and with zeros on all other positions.

We can also leave a_x, a_y, a_z out of the adjustment. Then the last column of the coefficient matrix of equation (28) disappears. The null hypothesis states now that there is no deformation. We test for a linear movement by using $\mathbf{Z}_\nabla$ in the test statistic of equation (34).

Generally the transformation between epoch i and $i-1$ is a similarity or congruence, not an affine transformation. Matrix $\mathbf{F}_i$ is constructed according to equation (20) from matrices $\mathbf{F}_{i,i-1}$ as given in equation (22) for the affine transformation. Matrix $\mathbf{Z}_f$ is the matrix that describes the constraints for a congruence or similarity transformation. The coefficient matrix of equation (13) or (14) is used to construct matrix $\mathbf{Z}_f$.

6 Adjustment and testing

6.1 Adjustment

System (28) is a linear system of observation equations and can be solved by least squares. If sufficient points are available in all epochs to determine the transformation parameters, the coefficient matrix is of full rank.

Because of the nonstochastic observations, and because of possible singularities of the covariance matrices of $\underline{\mathbf{a}}_1$ and the vectors $\underline{\mathbf{b}}_i$, the covariance matrix of the observation vector of system (28) is singular. To get a least squares solution of the system, at least five methods are available that make it possible to test nonstochastic observations in the same way as stochastic observations (Velsink, 2015b):

1. The adjustment model is split into two parts for the stochastic and the nonstochastic observations respectively, and a sequential adjustment is applied.
2. A switch is made from the model of observation equations to the model of condition equations.
3. The covariance matrix is regularised.
4. The standard deviations in the covariance matrix that are zero, are replaced by values that are very small.
5. The observations are orthogonalised and the nonstochastic observations eliminated. A follow-up adjustment determines the test quantities.

Because the system is linearised, iteration is needed to find the least squares solution. To start the iteration good approximate values for all observations and all parameters are needed, which have to satisfy the non-linear equations (10) and the non-linear constraints of section 5.3.2 or 5.3.3. As described by Velsink (2015a), in each iteration step the approximate transformation parameters are updated, using singular value decomposition. Also, in each iteration step, all $\underline{\mathbf{b}}_i$ and their covariance matrices $\mathbf{D}\{\underline{\mathbf{b}}_i\}$ are transformed with the new approximate transformation parameters to new coordinates $\underline{\mathbf{b}}'_i$ and $\mathbf{D}\{\underline{\mathbf{b}}'_i\}$ that are (for the common points) almost equal to $\underline{\mathbf{a}}_1$ and in the reference system of $\underline{\mathbf{a}}_1$.

In each iteration step the approximate values of all observations, and of all parameters have to comply again with the non-linear equations (10) and the non-linear constraints of section 5.3.2 or 5.3.3.

6.2 Deformation testing

If one of the five methods, mentioned in the previous section, is used, standard methods for testing can be applied with the formulas given by Velsink (2015b). Also the nonstochastic observations can be tested with the same formulas, which means that a method of testing deformation patterns is provided.

If it is not sure whether there is any deformation, or what type of deformation happens, a null hypothesis H_0 is formulated, where no deformation is assumed (∇ is missing in system (28)), and an alternative hypothesis H_a:

$$H_0 : E\{\underline{\Delta\mathbf{y}}\} = \mathbf{A}\,\Delta\mathbf{x}, \qquad (31)$$

$$H_a : E\{\underline{\Delta\mathbf{y}}\} = \mathbf{A}\,\Delta\mathbf{x} + \mathbf{Z}'_\nabla \Delta\nabla, \qquad (32)$$

where $\underline{\Delta\mathbf{y}}$, $\mathbf{A}$ and $\Delta\mathbf{x}$ are respectively the observation vector, the coefficient matrix and the parameter vector of system (28). In $\mathbf{A}$ the last column of the coefficient matrix is missing and in $\Delta\mathbf{x}$ the parameters $\Delta\nabla$. $\mathbf{Z}'_\nabla$ is the last column of the coefficient matrix of equation (28):

$$\mathbf{Z}'_\nabla = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{Z}_\nabla \end{pmatrix} \qquad (33)$$

The alternative hypothesis is tested against the null hypothesis, without the need to perform a complete adjustment of (32), by using test statistic $\underline{T}_q$ (Teunissen, 2006, p. 77):

$$\underline{T}_q = \frac{1}{\sigma^2} \underline{\hat{\mathbf{r}}}^T \mathbf{Z}'_\nabla (\mathbf{Z}'^T_\nabla \mathbf{Q}_{\hat{r}} \mathbf{Z}'_\nabla)^{-1} \mathbf{Z}'^T_\nabla \underline{\hat{\mathbf{r}}}. \qquad (34)$$

q is the number of columns in $\mathbf{Z}'_\nabla$ and gives the degrees of freedom of the test. σ^2 is the variance factor of unit weight, and $\underline{\hat{\mathbf{r}}}$ are the reciprocal least squares residuals as they follow from a weighted least squares adjustment (Velsink, 2015a) and for which holds, with $\underline{\hat{\mathbf{e}}}$ the usual least squares residuals and $\mathbf{Q}_y$ the cofactor matrix of the observations $\underline{\mathbf{y}}$:

$$\underline{\hat{\mathbf{e}}} = \mathbf{Q}_y\, \underline{\hat{\mathbf{r}}}. \qquad (35)$$

$\mathbf{Q}_{\hat{r}}$ is the cofactor matrix of $\underline{\hat{\mathbf{r}}}$. $\mathbf{Z}'_\nabla$ describes a testable deformation pattern, if the product $\mathbf{Z}'^T_\nabla \mathbf{Q}_{\hat{r}} \mathbf{Z}'_\nabla$ is a regular matrix.

The probability density function of $\underline{T}_q$ is a χ^2-distribution with an expected value of q. The test is to choose a significance level α, to compute the critical value and to test, whether the computed value of $\underline{T}_q$ exceeds the critical value. If this happens, the null hypothesis is rejected (Teunissen, 2006, p. 78).

6.3 S-basis invariance

In Velsink (2015b) and Velsink (2015c) it is shown that the test statistic of equation (34) is invariant for a change of S-basis of the parameter vector $\mathbf{x}$. It is evident from the fact that $\underline{\hat{\mathbf{r}}}$ can be computed from the model of condition equations, which is dual to the model of observation equations. In this dual model the parameter vector $\mathbf{x}$ has been eliminated, and therefore a change of S-basis of $\mathbf{x}$ doesn't influence $\underline{\hat{\mathbf{r}}}$.

The test statistic of equation (34) is also invariant for changes of S-bases of the observed coordinate vectors $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_i$, i=2, ..., p, if deformation patterns are tested. To see this, model (28) is simplified and reduced. To do this, we assume all observed vectors $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_i$, and also the parameter vectors $\mathbf{c}_i$ to contain coordinates of the same points in the same order, from which follows:

$$\mathbf{P} = \text{unit matrix.} \tag{36}$$

We also assume stability of all points, and therefore:

$$\Delta\mathbf{c}_i = \Delta\mathbf{c}_j, \text{ with } i,j = 1, \ldots, p, \tag{37}$$

and we reduce $\Delta\mathbf{c}$ to a vector $\overline{\Delta\mathbf{c}}$ with the coordinate parameters of only one epoch. With matrix $\mathbf{I}_p$ defined with unit matrices $\mathbf{I}$ as:

$$\mathbf{I}_p = \left(\mathbf{I}, \ldots, \mathbf{I}\right)^T, \tag{38}$$

we get:

$$\Delta\mathbf{c} = \mathbf{I}_p\,\overline{\Delta\mathbf{c}}. \tag{39}$$

This means that the nonstochastic observations $\mathbf{z}_d$ disappear. Furthermore we assume that the nonstochastic observations $\mathbf{z}_f$ are elimated. This can be done by noting that the equation:

$$\mathbf{0} = \mathbf{Z}_f \Delta\mathbf{f} \tag{40}$$

means that $\Delta\mathbf{f}$ lies in the nullspace of $\mathbf{Z}_f$. If $\mathbf{N}$ is a base matrix that spans this nullspace, we have:

$$\Delta\mathbf{f} = \mathbf{N}\overline{\Delta\mathbf{f}}, \tag{41}$$

with $\overline{\Delta\mathbf{f}}$ a vector of coefficients, which can be used as the new vector of unknown transformation parameters. If, for example, $\Delta\mathbf{f}$ contains 12 parameters of an affine transformation and there are 5 nonstochastic observations to constrain the transformation into a similarity transformation, $\overline{\Delta\mathbf{f}}$ contains 7 transformation parameters. With the definition $\overline{\mathbf{F}} = \mathbf{FN}$, it follows that:

$$\mathbf{F}\Delta\mathbf{f} = \overline{\mathbf{F}}\,\overline{\Delta\mathbf{f}}. \tag{42}$$

So if we use $\overline{\mathbf{F}}\,\overline{\Delta\mathbf{f}}$ instead of $\mathbf{F}\Delta\mathbf{f}$ in model (28), we can omit the nonstochastic observations $\mathbf{z}_f$.

Because of the stability assumption, no parameters ∇ exist and no matrix $\mathbf{Z}_\nabla$.

With (36), (39) and (42), model (28) is written as:

$$E\{\underline{\Delta\mathbf{b}}\} = \mathbf{I}_p\,\overline{\Delta\mathbf{c}} - \overline{\mathbf{F}}\,\overline{\Delta\mathbf{f}} \tag{43}$$

To eliminate $\overline{\Delta\mathbf{c}}$, we define matrix $\mathbf{H}$ as:

$$\mathbf{H} = \begin{pmatrix} -\mathbf{I} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \ldots & \ldots & -\mathbf{I} & \mathbf{I} \end{pmatrix}^T, \tag{44}$$

and the vector $\mathbf{d}$, containing the difference vectors of all epoch intervals, as:

$$\mathbf{d} = \mathbf{H}^T\mathbf{b}. \tag{45}$$

Premultiplying equation (43) with $\mathbf{H}^T$, we get:

$$E\{\underline{\Delta\mathbf{d}}\} = -\mathbf{H}^T\,\overline{\mathbf{F}}\,\overline{\Delta\mathbf{f}}. \tag{46}$$

This model has the same redundancy as model (43) and yields the same least squares solution.

Let the vectors $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_i$, $i = 2, \ldots, p$ all have been S-transformed to other S-bases. It means that we have new vectors $\underline{\mathbf{a}}'_1$ and $\underline{\mathbf{b}}'_i$, taken together in vector $\underline{\mathbf{b}}'$:

$$\underline{\mathbf{b}}' = \underline{\mathbf{b}} + \mathbf{S}\underline{\boldsymbol{\psi}}, \tag{47}$$

where $\underline{\boldsymbol{\psi}}$ is the vector of the differential transformations of the coordinate vectors of all epochs. These relate, however, to the same degrees of freedom as the transformations in $\overline{\Delta\mathbf{f}}$. This means that we can take $\mathbf{S} = \overline{\mathbf{F}}$.

A proof for two epochs that test statistic (34) is invariant for changes of S-bases of $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_2$, by proving that $\underline{\hat{\mathbf{r}}}$ and $\mathbf{Q}_{\hat{r}}$ are invariant, is given by (Velsink, 2015c). The extension to more than two epochs is possible by using reduced model (46). This model can be solved by switching to the model of condition equations with matrix $\mathbf{G}$, which is chosen to fulfil:

$$\mathbf{G}^T\ \mathbf{H}^T\overline{\mathbf{F}} = \mathbf{0}, \tag{48}$$

with $R(\mathbf{G})$ the complementary space of $R(\mathbf{H}^T\overline{\mathbf{F}})$. It follows with the same reasoning as given by Velsink (2015c) that $\underline{\hat{\mathbf{r}}}$ and $\mathbf{Q}_{\hat{r}}$, as they follow from solving model (46), are invariant for changes in S-bases of $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_i$, $i = 2, \ldots, p$.

The conclusion is that if hypotheses concerning deformation patterns are formulated in terms of the original model (28), and they can be reformulated in terms of model (46), which is generally possible, test statistic (34) is invariant for changes to other S-bases of the coordinate vectors $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_i$, $i = 2, \ldots, p$.

A deformation hypothesis may concern a point that is part of the S-basis definition and whose coordinates are fixed with a zero standard deviation. No S-transformation is needed to test such a point for deformation. It is demonstrated by the example of Velsink (2015b).

7 Experimental validation

The proposed model can be applied to the 3D monitoring by GPS and total stations of deformations of buildings, harbour quays, bridges, tunnels, land slides, etc. The model gives the possibility to compute statistics and to test hypotheses that describe complex deformation patterns, like the abnormal movement of a subset of points through many epochs, or the periodic oscillation of a subset of points, for example caused by changes of temperature.

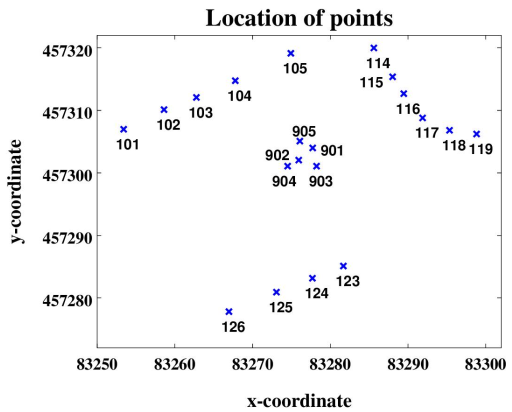

Fig. 2: 15 object points, 5 instrument points

To validate experimentally the model, the monitoring of some buildings is taken. To be able to judge effectively the performance of the model, observations have been generated with known standard deviations, to which artificially deformations have been added. Fifteen points have been measured with direction and distance observations from a total station during five epochs. The fifteen points are positioned on three buildings (figure 2), which are monitored because of construction works. The instrument point is not fixed (not monumented). The observations are adjusted using the software package MOVE3 (www.move3.com), resulting in x, y, z coordinates and their covariance matrix. The network is not attached to a control network.

A Matlab programme has been written to do the computations. The observations have been generated with the following standard deviations:

- directions: 0.3 mgon;
- distances: 1 mm;
- zenith angles: 0.3 mgon.

The precision with which a point is defined (idealisation precision) is supposed to be 0.5 mm, indicating the precision by which a removable prism can be put on a point.

First no deformation is put in the observations. The adjustment model to test stability of all points is created by adding for each epoch interval, for each point and for each coordinate direction a nonstochastic observation, i.e. $4 \times 15 \times 3 = 180$ nonstochastic observations:

$$\begin{cases} \mathbf{0} = \Delta\mathbf{c}_1 - \Delta\mathbf{c}_2 \\ \mathbf{0} = \Delta\mathbf{c}_2 - \Delta\mathbf{c}_3 \\ \mathbf{0} = \Delta\mathbf{c}_3 - \Delta\mathbf{c}_4 \\ \mathbf{0} = \Delta\mathbf{c}_4 - \Delta\mathbf{c}_5 \end{cases} \tag{49}$$

From these equations follows matrix $\mathbf{Z}_d$ in equation (28).

The epochs are joined together with similarity transformations, which are realised for each epoch interval by five constraints on the affine transformation parameters.

The model contains 425 observations ($5 \times 15 \times 3 = 225$ coordinates, $4 \times 5 = 20$ transformation constraints, 180 point constraints), and 273 parameters (225 coordinates, $4 \times 12 = 48$ transformation parameters), which yields a redundancy of 152. Adjusting the model leads to an overall model test of 0.60. With a critical value of 1.004, based on the use of the B-method of testing with a significance level of a one-dimensional test of 0.1% and of 47% for the overall model test, the null hypothesis is accepted.

Table 1: Test of linear point movement over 5 epochs; values are per epoch interval

	Stat.	Est. def. (mm)		
Pnt.		x	y	z
101	70.9	0.9	1.1	1.0
103	16.9	-0.4	-0.5	-0.3
102	7.0	-0.1	-0.1	-0.5
104	4.7	-0.2	-0.3	-0.1

Then a movement of 1 mm in each epoch interval, in the direction of each coordinate axis of one point (point 101), is added and observations are generated. The same adjustment model as before is used and leads to rejection of the null hypothesis. The movement is then modelled with twelve nonstochastic observations. Assume vector $\mathbf{c}_{101}^{(i)}$ is the subvector of vector $\mathbf{c}$ that contains the x, y and z coordinates of point 101 in epoch i. Let vector $\mathbf{a}$ contain the movements in x, y and z direction between epoch i and j, and $\Delta\mathbf{a}$ the difference of $\mathbf{a}$ with its approximate value, necessary for the linearised model. The following 12 nonstochastic observations describe the deformation.

$$\begin{cases} \mathbf{0} = \Delta\mathbf{c}_{101}^{(1)} - \Delta\mathbf{c}_{101}^{(2)} + \Delta\mathbf{a} \\ \mathbf{0} = \Delta\mathbf{c}_{101}^{(2)} - \Delta\mathbf{c}_{101}^{(3)} + \Delta\mathbf{a} \\ \mathbf{0} = \Delta\mathbf{c}_{101}^{(3)} - \Delta\mathbf{c}_{101}^{(4)} + \Delta\mathbf{a} \\ \mathbf{0} = \Delta\mathbf{c}_{101}^{(4)} - \Delta\mathbf{c}_{101}^{(5)} + \Delta\mathbf{a} \end{cases} \tag{50}$$

From these equations the matrix $\mathbf{Z}'_\nabla$ of equation (32) is deduced, and the alternative hypothesis tested against the null hypothesis. The same test is used to test for linear movement of all other points. The deformed point shows the largest value of the test statistic (table 1 under "Stat."), with a critical value of 12.6 and a significance level of 0.6 %. The estimated deformation (Velsink, 2015c, eq. (42)) in each epoch interval is given in the same table for point 101 and three other points with large test statistics. The estimated deformation of point 101 resembles closely the values that have been put intentionally into the coordinates, and the length of the deformation vector is even the same: 1.7 mm in each epoch interval. Table 2 gives the minimal detectable deformations as the lengths of the semi-axes of the ellipsoid determined by (Velsink, 2015c, eq. (44)):

$$\sigma^2\lambda_0 = \nabla_0^T \mathbf{Z}'^T_\nabla \mathbf{Q}_{\hat{r}} \mathbf{Z}'_\nabla \nabla_0, \tag{51}$$

with λ_0 the non-centrality parameter of the χ^2-distribution and ∇_0 describing the minimal detectable deformations. They give the deformations that can be detected with the three-dimensional point test of five epochs with a power of 80%.

Table 2: Minimal detectable deformations (m.d.d.); values are per epoch interval

	M.d.d. (mm)		
Pnt.	axis 1	axis 2	axis 3
101	1.55	0.80	0.76
103	1.49	0.73	0.73
102	1.52	0.76	0.74
104	1.48	0.72	0.72

Finally five points (101,...,105) are given a movement of 1 mm in both the x and y direction and -0.7 mm in the z direction in each epoch interval. It is modelled by 60 nonstochastic observations. Let vector $\mathbf{c}_{101-105}^{(i)}$ be the subvector of vector $\mathbf{c}$ that contains the x, y, z coordinates of the five points in epoch i. Let $\mathbf{k} = (1, 1, 1, 1, 1)^T$, $\mathbf{I}_3$ the (3×3)-unit matrix, and $\mathbf{E} = \mathbf{I}_3 \otimes \mathbf{k}$, with $\otimes$ denoting the kronecker product. From the following nonstochastic observations the matrix $\mathbf{Z}_\nabla$ is deduced.

$$\begin{cases} \mathbf{0} = \Delta\mathbf{c}_{101-105}^{(1)} - \Delta\mathbf{c}_{101-105}^{(2)} + \mathbf{E}\,\Delta\mathbf{a} \\ \mathbf{0} = \Delta\mathbf{c}_{101-105}^{(2)} - \Delta\mathbf{c}_{101-105}^{(3)} + \mathbf{E}\,\Delta\mathbf{a} \\ \mathbf{0} = \Delta\mathbf{c}_{101-105}^{(3)} - \Delta\mathbf{c}_{101-105}^{(4)} + \mathbf{E}\,\Delta\mathbf{a} \\ \mathbf{0} = \Delta\mathbf{c}_{101-105}^{(4)} - \Delta\mathbf{c}_{101-105}^{(5)} + \mathbf{E}\,\Delta\mathbf{a} \end{cases} \tag{52}$$

with $\Delta\mathbf{a}$ as defined before.

The null hypothesis is rejected again. The test of the hypothesis that the five points have shifted gives a very large test statistic (74.2 with a critical value of 12.6, if the significance level is 0.6 %), indicating that it is a very good hypothesis. The estimated deformation and the minimal detectable deformations are given in table 3. The length of the deformation vector is 1.6 mm, which is exactly the length of the vector that has been put intentionally into the coordinates.

Point movements that are nonlinear in time are modelled by nonstochastic observations that are nonlinear functions of the deformation parameters. To be used in the model, the functions have to be linearised.

If the deformation pattern to be expected is not known, a search has to be performed for the best alternative hypothesis. A strategy is described by Velsink (2015b) for two epochs. Extending it to more than two epochs, one could for example systematically test for a constant linear movement through all

Table 3: Linear movement of points 101-105 over five epochs; values are per epoch interval

	Est. def. (mm)			M.d.d. (mm)		
Pnt.	x	y	z	axis 1	axis 2	axis 3
101 - 105	1.0	0.7	-1.1	0.93	0.71	0.60

epochs of each point individually, of combinations of two points close together, of combinations of three points close together, etc. Because it is not needed to solve a complete adjustment model, only to compute test statistic (34), its degrees of freedom q determines the computational burden of testing many hypotheses.

8 Conclusions

A model has been built for the adjustment of a time series of 3D coordinates in a geodetic point field. The covariance matrices of the coordinates of all epochs of the time series are used and they may be full and singular. Deformation patterns, or their absence, are modelled as nonstochastic observations. To make the testing of the model invariant for S-transformations, transformations between all epochs are built into the model. The transformations can be similarity or congruence transformations, and are modelled as affine transformations, subject to constraints. The constraints are implemented as nonstochastic observations. The model is first built as a nonlinear one, and then linearised. The approximate parameter values and their updates in the iteration steps (needed because of the linearisation) have to comply with all nonstochastic observations. For the rotation parameters this is accomplished with singular value decomposition.

In many cases it is a sound deformation analysis procedure to formulate a null hypothesis that assumes no deformation. The nonstochastic observation equations state that the coordinate differences between the epochs are expected to be zero after the transformations. Alternative hypotheses are formulated that describe movements of one or many points over one or many epoch intervals. Standard hypothesis testing is used to test the alternative hypothesis against the null hypothesis. The quality of the tests is described by the sizes of the minimal detectable deformations.

The point movements are formulated as nonstochastic observation equations, which give the matrices to be used in the testing equations.

The model and its adjustment and testing have been verified experimentally with a geodetic network, where 15 points are measured by a total station during five epochs. The results show that 3D deformation analysis of time series of coordinates is possible with the model proposed.

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